KOLAM DESIGNS BASED ON 'VERSATILES'

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A B S T R A C T

Versatile is a cyclic quadrilateral. Used as a tile, its multiple copies can be assembled into different polygonal shapes. By inscribing a *kolam* design into the versatile the polygons transform into *kolam* motifs. The versatile is a unique shape that can tile 13 different polygons because one of its angles is 60°. By embedding a discrete square grid of dots, the assembled shapes are restricted to 4-sided figures: a square, a square with a hole, a rectangle with a hole and a parallelogram. 'Versatile *kolams*' of size 10 x 10, 12 x 12, 8 x 14, 9 x 12, 9 x 9 and 9 x14 are presented, the last one being close to a Golden Rectangle. If the defining angle of the versatile is different from 60° – in the range 45° to 90° – the motifs can only be 4-sided. The generalized versatile is called a 'Universatile' (U-tile). It is suitable for *kolam* motifs. As an illustration 4 assembled motifs of a U-tile with angle 75° are shown.

The well-known puzzle of missing or vanishing square related to the Fibonacci Series (0 1 1 2 3 5 8 13 ....) has a counterpart in U-tiles. The origins of the two fallacies are compared and notable differences are found.

Finally, brief reference is made to the general tiling problem in 'mathematical logic' and its relation to Gödel's theorem on undecidable problems in arithmetic.

....

"It seems a very ordinary quadrilateral" I said.

"No Watsup, it is a very extraordinary quadrilateral ...."

....

"(Soames), you think that copies of this one tile (quadrilateral) can form all 13 motifs?"

"I am certain of it .... It is a remarkably versatile – er – tile ..... Thanks to its cunning geometry" he replied.

[From *"Professor Stewart's Casebook of Mathematical Mysteries:* 'The Enigma of the Versatile Tile', p 195, 292] [1]

1. INTRODUCTION.

The quadrilateral alluded to in the above conversation is shown in Figure 1. One pair of opposite angles A and B are both right angles. Angle D is 60^o and it fixes angle C as 120^o. The quadrilateral ACBD is cyclic – i.e. all the 4 vertices lie on a circle with CD as diameter. The sides AC and AD are equal.

In [1] Professor Stewart demonstrates how multiple copies of the quadrilateral 'tile' can be arranged to form 13 different polygons. The polygons include *n*-sided figures: equilateral triangles (n = 3), squares, rectangles, parallelograms, trapeziums (n = 4), pentagon (n = 5), hexagon (n = 6) and dodecagons (n = 12).

In Recreational Mathematics dealing with tiling problems in general, it is common to embed shapes of different types into the tile. Here I consider embedding into the tile a *kolam* design. When copies of such a tile are arranged to form different polygons, they form different *kolam* motifs. The *kolam* is drawn on a square grid of dots. Embedding a square grid in a tile with acute and oblique angles (60^o and 120^o) allows only certain motifs such as squares and rectangles. In Figure 2 are shown 4 of them: a square with a hole, a square, a rectangle with a hole and a parallelogram each made up of 4 tiles.

Tiling a plane (without gaps or overlaps) with polygons of different shapes is a branch of mathematics with profound implications for 'mathematical logic' and the solvability or otherwise of problems [1].

The Versatile-based *kolams* define a new genre of *kolam* designs distinct from the Fibonacci *Kolams* based on Fibonacci Recurrence. The latter are coded by a quartet of integers Q ($a\ b\ c\ d$) in which a, b are free to choose. By Fibonacci Recurrence $c = a + b$, $d = c + b$. The numbers are related by the identity

$$d^2 = a^2 + 4\, b\, c$$

A geometric interpretation of the above is the following: 'a square $d \times d$ contains a smaller square $a \times a$ at the centre and 4 rectangles $b \times c$ fill the annular space in a cyclic arrangement'. By embedding *kolams* into them one generates Fibonacci *Kolams* (FK in short). A square FK of any desired size can be achieved by proper choice of a and b.

Fibonacci Recurrence can be extended to rectangles too, but one requires 2 quartets Q1 and Q2. FK's are developed in a series of articles in 5 parts in [2]. An overview is given in [3].

2. VERSATILE *KOLAM* DESIGNS.

The Versatile in Figure 1 (right side) has all the sides and angles labelled. The two equal sides at right angles are assigned value 1 (unit) to fix the scale of the entire tile. Side BD is drawn at an angle φ to the diagonal DC and CB is perpendicular to DB. In the right-angled triangle BCD,

$$CD = \sqrt{2} \qquad BD = \sqrt{2}\cos\varphi \qquad BC = \sqrt{2}\sin\varphi$$

Angles $\quad A = B = 90^o \qquad D = 45 + \varphi \qquad C = 135 - \varphi$

A versatile is fully determined by the angle φ. For $\varphi = 15^o$

$$AC = 1 \quad AD = 1 \quad BD = (\sqrt{3} + 1)/2 = 1.366 \quad BC = (\sqrt{3} - 1)/2 = 0.366.$$

$$A = B = 90^o \quad D = 60^o \quad C = 120^o$$

In Figure 2, the versatile is shown in the left in top row. Four copies of this tile are arranged to form the motifs: square with a hole, a square (top row), rectangle with a rectangular hole and a parallelogram (bottom row). For all the motifs $\varphi = 15^o$. They are adapted from [1].

I propose to recreate the above motifs with *kolam* designs. The *kolam* grids are discrete with integral number of dots whereas the motifs are not. It turns out that each *kolam* motif has identical versatiles whereas the versatile can differ slightly from one motif to another. Since the versatile is determined by φ, its value differs slightly from 15^o across different motifs. Figure 2 is duplicated in Figure 3 and the relevant measures of the motifs are labelled. They are also tabulated in Table 1.

How does one design a versatile to begin with? The easiest way is to decide on a square of some size and divide it into 4 identical versatiles. This is illustrated by the following example.

2.1. Versatile *Kolam*: square 10 x 10.

The *kolam* is shown in Figure 4. The way it is assembled is described below. Consider the square grid without the *kolam*. Each side has 10 dots, but only 9 unit cell lengths. Divide each side into two parts of length 2 and 7 in a cyclic manner and mark the dividing points. Join two points on opposite sides drawing two lines intersecting at the centre. They divide the square into 4 identical versatiles. The thin lines – 4 along the sides and 2 inclined lines – delineate them. The angle φ of the versatile is determined as follows. The two parts of each side of the square are in the ratio $\sqrt{2}\sin\varphi$: $\sqrt{2}\cos\varphi$ or $\tan\varphi$ (Figure 3c). Setting $\tan\varphi = 2/7$, $\varphi = \arctan(2/7) = 15.95^o$. The angle of the tile $\theta = 45 + \varphi = 60.95^o$.

Notice that the inclined lines cut through the *kolam* shapes at some places. They are the places where a unit cell is shared by adjacent versatiles. In Figure 4 on the right is shown the tile as oriented in the third quadrant. Each tile has 25 unit cells of which 5 are shared with other tiles. The shared cells can be suitably modified to create a stand-alone versatile. This versatile can be used in forming the other motifs of Figures 2 and 3.

2.2. Versatile *Kolam*: square with a hole 12 x 12.

In Figure 5 a 12 x 12 square grid is divided into 4 identical versatiles leaving an empty square at the centre. From the mid-point of each side a line is drawn at an angle of 60^o, such that 4 lines enclose a square at the centre, the 'hole'. Each versatile is rotated 90^o to the adjacent tile. The tile in the second quadrant is shown on the right side of Figure 5. A *kolam* very similar to the one in Figure 4 (right) is embedded in the versatile. Four such versatiles are put together to form 2 loops. It cannot be reduced to a single loop as explained in Section 5. Each versatile has 29 units cells. The four tiles together have 116 (4 x 29) cells. The hole accounts for the remaining 28 (144 – 116) cells.

2.3. Rectangle 8 x14 with a rectangular hole.

The first task is to select a suitable rectangle following the template of Figure 3(d). The longer side (the width w) is to be divided into two parts in the ratio $1 : \sqrt{2} \cos \varphi$ and the shorter side (the height h) in the ratio $1 : \sqrt{2} \sin \varphi$. For $\varphi = 15^o$ the ratios are 1 : 1.366 and 1 : 0.366 respectively. The first ratio is 0.732 which is very close to 11/15 (=0.733). A length of 13 cells units in w can be divided into parts 5.5 and 7.5 (half-integral lengths are easily marked as the *kolam* lines intersect exactly between two adjacent dots). So w is chosen as 14 (# of dots). The height h is to be chosen such that the ratio w/h is close to the ratio of sides in the template. The value is 8 (# of dots). The 7 cell units are divided in the proportion 1.5 : 5.5. The *kolam* is shown in Figure 6. From the points dividing the sides, lines are drawn at 60^o to the side so that they enclose a rectangle at the centre as in Figure 3(d). Thin lines connecting the dots on the 4 sides along with the inclined lines demarcate the versatiles.

The 4 versatiles in Figure 6 are not the same. The pair in quadrants 1 and 3 (Q1 and Q3) are the same and the pair in quadrants 2 and 4 (Q2 and Q4) are the same so that two-fold rotational symmetry is ensured. The versatiles in Q3 and Q2 are shown on the right. A careful accounting of the number of cells shows that there are 16 shared cells and the remaining 96 (8 x 14 – 16) are in the other cells in the tiles and the hole. Whereas a single tile was enough to create all the motifs of Figure 2, including the rectangle, it requires 2 different tiles in the rectangular *kolam* of Figure 6 because of the discrete nature of the *kolam* grid.

The number of loops is 2 just as in the case of the square with a hole (Figure 5).

2.4. Parallelogram 9 x 12.

The template for the composition of 4 versatiles into a parallelogram is shown in Figure 2(e). The ratio $h/w = 1.5/2 = 3/4$ (Table 1). The grid size is 9 x 12 (Figure 7) [4]. The acute angle is 60^o so that $\varphi = 15^o$. The grid can be divided into 4 identical versatiles, one of which is shown on the right side.

The stacking of the 4 tiles side by side is an unusual construction. It leaves no hole. They are all not the same. The middle two tiles are the same each with 24 cells and they share the central tile. The two remaining tiles have each 26 cells. All the four account for 101 cells. This is the number of dots in the grid, 11 in each row with an extra dot in the top and bottom rows. The number of loops is 3.

2.5. A Golden Rectangle 9 x 14.

The Golden Rectangle is related to the Fibonacci Series

0 1 1 2 3 5 8 13 21 34 55..........

in which every number after the first two is the sum of the two preceding numbers [5]. The ratio of 2 consecutive numbers

3/2 5/3 8/5 13/8 21/13 34/21 55/34

converges to 1.61803 ....called the Golden Ratio (GR). Rectangles with sides in GR are Golden Rectangles. Here the chosen ratio is 13/8 (= 1.625) that differs from GR by < 0.007 or 0.43 %. The grid size is 9 x 14 (# of dots) so that the side length $h = 8$ and $w = 13$. What is the value of φ for the versatile to be used to construct the above rectangle with 4 identical tiles as per Figure 3(d)? From Table 1

$$w = 1 + \sqrt{2}\cos\varphi \qquad h = 1 + \sqrt{2}\sin\varphi$$

Setting w/h as 13/8 and simplifying

$$8\cos\varphi - 13\sin\varphi = 3.536$$

Solving for φ $\qquad \varphi = 18.22^{o}$

In Figure 8 the side w is divided in proportion 5.5 : 7.5 and side h as 2.5 : 5.5. From the marked points lines are drawn at an angle $\theta = 45 + \varphi = 63.2$ to the sides so that they enclose a rectangular 'hole' at the centre. The versatiles in Q1 and Q2 are shown on the right. Two of each are placed as shown to make a 9 x 14 *kolam* with a rectangular hole. Tiles Q1 and Q2 are nearly the same each with 30 unit cells. The remaining 6 cells (126 – 120) are accounted by the hole. The number of loops is 2.

2.6. Versatile *Kolam* 9 x 9.

In Figure 9 the side of length 8 units is divided in the ratio 2:6. The dividing points on opposite sides are joined to create 4 identical versatiles. The angle φ is arctan (2/6) = arctan (1/3) = 18.26°. The versatile is fully determined by φ. The tile in Q2 is shown on the right. Four such tiles are placed cyclically in the square and spliced together. Each tile has 20 unit cells. In addition there is a 4-way splice at the centre. The total number of unit cells is 81.

Notice that φ is 18.26° which is very close to 18.22° that characterises the versatile in the 9 x 14 Golden Rectangle (Figure 8). This implies that the two versatiles are nearly of the same shape and differ only in absolute scale.

3. BEYOND VERSATILES - THE 'UNIVERSATILES' (U-TILES)

By definition the versatile has φ = 15 deg. Four versatiles are arranged to form different motifs [1]. In embedding discrete *kolams* into versatiles, the angle φ had to be tweaked slightly – up to 18.26° for 9 x 9 (Figure 9, Section 2).

It is worth emphasizing that there is no approximation in the calculation of φ for Figures 4 to 9. The ratios are chosen as simple fractions and the corresponding φ values are determined according to measures given in Table 1 and Figure 3. Indeed the versatiles can be extended to take any value of φ in the range 0° – 45°. We call them **Universatiles** (U-tiles for short). In the present context of embedding *kolams* into U-tiles, the most relevant φ values are those that allow tan φ, $\sqrt{2}$ cos φ and $\sqrt{2}$ sin φ to be equal to simple fractions. In Table 2 several such are tabulated to cover the range of φ 10° to 45°.

To illustrate the use of Table 2 for the U-tile with φ = 30° : Four of them are assembled to produce 4 motifs as in the case of φ = 15° in Figure 2. From Table 2 note that arctan (4/7) is 29.74° only 0.26° (< 1 %) short of 30 . The simplest way to realize the U-tile is to divide a square of length 11 units into 4 identical U-tiles similar to Figure 9. The number of dots on each side is 12, so that the *kolam* grid is 12 x 12 (dots). The sides are divided in proportion 4 : 7 cyclically (just as in Figure 9 where the proportion is 2 : 6). Figure 10 shows the four assembled motifs.

For the versatile (φ = 15°) 4 *kolam* motifs were presented in Figures 4 to 7. In embedding a discrete grid of dots in the tile, slightly different tiles are used for different motifs, but in each motif the tiles are the same. For U-tile (φ = 30°) *kolam* motifs similar to Figures 4 to 7 can be drawn using the ratios $\sqrt{2} \cos \varphi$ and $\sqrt{2} \sin \varphi$ given in Table 2.

The motif 'square with a square hole" lends itself to an interesting puzzle, a mathematical fallacy akin to the fallacy of the missing square related to the Fibonacci Series.

4. THE FALLACY OF THE 'MISSING SQUARE'.

In a popular puzzle related to the Fibonacci Series

0 1 1 2 3 5 8 13 21 34 55..........

a 8 x 8 square is dissected into 4 parts and reassembled into a rectangle 5 x 13. The two areas differ by one unit (65 – 64) which is not easily visible to the naked eye. (See Figure 11a bottom) [5]. The missing area is actually a slim parallelogram of angle 1.25^{o}. The trick works for any consecutive triplet (a b c) – like (3 5 8), (5 8 13), (8 13 21) – since

$$b^2 - a c = +1 \text{ or } -1$$

with + and – signs alternating as the triplet 'advances' e. g. $5^2 - 3 \times 8 = +1$, $8^2 - 5 \times 13 = -1$, $13^2 - 8 \times 21 = +1$. The angle of the parallelogram approaches 0. This is the fallacy of the 'vanishing' square.

On the top of Figure 11(a) is shown the partition of a 8 x 8 square into 4 U-tiles and their re-assembly into a square with a hole of area ≈ 4 units, too large to be invisible. However by suitable choice of the U-tile, one can make the hole as small as desired. For example by dividing the side of a square (8 x 8) in proportion 4.5 : 3.5 or 9 : 7 (instead of 5 : 3) the central hole is reduced to one unit (Figure 11b). This is a U-tile version of the fallacy of the vanishing square.

How is the missing area accounted for? In Figure 11(b) the two squares look identical (8 x 8) but they are not. The right square side is 8.063 units. The difference in the two areas is one unit which is the area of the hole. The missing hole area is therefore spread out as a narrow strip of width 0.063 around the 4 sides of the square.

A closer analysis of the genesis of the missing hole in the Fibonacci and U-tile versions reveals notable similarities as well as differences. In Fibonacci (Figure 11a bottom) a square is reassembled as a *rectangle* whereas in the U-tile (Figure 11b) a square is reassembled as a *square with a hole*. In Fibonacci, as the size of the square is increased, the missing area remains one unit. The ratio of the missing area and the square area is ($1/F_n^2$) where F_n is the *n*th Fibonacci number (F_0 =0, F_1 = 1, F_2 = 2 ......). In contrast in the U-tile, the square remains fixed and the hole area decreases as the φ value of the U-tile increases. Let p, q be the sides of the square and the hole. From Figure 3(b) and Table 1

$$p = 2 \qquad q = 2\cos(45 + \varphi) = 2\sin(45 - \varphi)$$

The ratio $q/p = \sin(45 - \varphi)$. The ratio of the areas R,

$$R = (q/p)^2 = \sin^2(45 - \varphi)$$

To determine φ note that $\tan\varphi$ is the proportion in which the sides of the square are divided. In Figure 11(b) this ratio is 3.5 : 4.5. $\tan\varphi = 3.5/4.5 = 7/9$ and $\varphi = \arctan(7/9) = 37.87^o$.

$$R = \sin^2(45 - \varphi) = \sin^2(7.13) = 0.0154$$

[Check: hole area $q^2 = p^2\,(0.0154) = 64(0.0156) = 1$ unit]. A good approximation to $\sin\alpha$ for $\alpha < 10^o$ is α in radians or ($\alpha/57.3$). Putting $45 - \varphi = \delta$

$$R \infty \delta^2$$

R tends to 0 as δ tends to 0, quadratically.

Comparing the rates of decline of hole area in the two cases:

(1) In Fibonacci case the decline is inversely proportional to F_n^2. A very good approximation to F_n even for low *n* is

$$F_n \approx (GR)^n / \sqrt{5}$$

[6]. The ratio

$$R = 1/F_n^2 \infty (GR)^{-2n} \infty (2.618....)^{-n}$$

declines exponentially as *n*.

(2) In the U-tile, R declines as δ^2, a power law.

Exponential and power law functions are two major classes in statistics of growth and decay. Typically exponential decay is much faster than the power law. The latter tends to have longer tails, It is interesting that both kinds of decline occur in the vanishing hole.

5. DISCUSSION AND CONCLUSION.

The FK's are built from 8 basic shapes, each inscribed in a square tile. They are called A B C D E F G H (Figure 12) [2]. Rotations and reflections yield 64 orientations of which only 31 are distinct (A1 A2 A3.....B1 B2 B3... ......H1 H2 H3....). The U-tile *kolams* are based on the same 8 shapes. In FK's the basic modules are rectangles and squares in which the shapes are embedded. In U-tile *kolams* the basic module is a cyclic quadrilateral. The modules are put together as in a jigsaw puzzle. They are 'stitched' together following some splicing rules that are akin to the matching rules in tiling problems. These rules are the same in FK and the U-tile *kolams*.

A cardinal requirement of all *kolams* is symmetry: 4-fold for squares and 2-fold for rectangles and parallelograms. Generally the *kolam* is required to have a single loop. This was the criterion adopted for FK's. In U-tile *kolams* however, the squares 10 x 10, 9 x 9 (Figures 4, 9) are single loop whereas the rectangles (Figures 5, 6, 8) are 2 loops. The parallelogram (Figure 7) has 3 loops; with two additional splices the number may be reduced to a single loop. The parity of the number of loops (odd/even) is discussed in great detail in 'Evolution Loops' (Part IV in ref [2]) [7].

Like the U-tile *kolams* the FK's too can be viewed as tiling problems. For example, given 4 copies of a rectangle $b \times c$ ($c > b$), how can they be fit inside a square leaving a smaller square at the centre so that the whole pattern has 4-fold symmetry? In Figure 13, four rectangles are arranged in a cyclic manner such that the big and small squares have sides $c + b$ (= d) and $c - b$ (= a). The numbers (a b c d) form the Fibonacci Quartet Q (a b c d).

FK's are not restricted to the canonical Fibonacci series 0 1 1 2 3 5.......... An FK of any desired size (square or rectangle) can be constructed by using the Generalized FQ: Q (a b c d) in which a and b are free to choose and $a = c - b$, $d = c + b$ [2, Part II] .

The U-tiles too can be adapted to create the 4 motifs of Figure 10 in any desired size. The versatile is an U-tile with $\varphi = 15^{o}$ and is unique in that it can be used to tile the plane into 13 types of polygons [1]. In U-tiles φ can take any value 0 to 45 but the polygons are restricted to squares, rectangles and parallelograms. So for such motifs, U-tiles offer a wider choice.

As mentioned in the Introduction, the general problem of filling a plane with polygons of different shapes without any gap or overlap – the tiling problem – in mathematics has profound implications for mathematical logic [1] [5].

The tiling of a plane can be periodic or aperiodic (non-periodic). In a periodic tiling a bounded region repeats itself - *without rotation or reflection* - in two different directions indefinitely. The only regular polygons that can do this are the equilateral triangle, the square and the hexagon. In an aperiodic tiling there is no fundamental region that repeats itself. A regular hexagon can only tile periodically. But there are many shapes that tile both periodically and aperiodically. A periodic tiling like a chess-board can be converted into an aperiodic tiling by dissecting each square into 4 versatiles as in Figure 3c and changing their orientations to prevent periodicity. The same is true for the U-tiles in Figure 10 (top right).

Is there a set of tiles that can tile the plane aperiodically but not periodically ? This is the question with profound implications in 'logic'. More specifically, given a finite number of tiles is there an algorithm that can provide an answer whether or not they can tile the plane. The answer to the question is: '*there is no such algorithm'*. In 1931 Kurt Gödel proved that there exist problems in arithmetic for which no algorithm can decide whether they are true of false. Gödel's example of such a problem was rather esoteric. However, the tiling problem is a natural example of such an undecidable proposition. See [1] for details.

The Fibonacci *Kolams* and the U-tile *kolams* 'look' very similar in a broad aesthetic view because they are based on the same shapes, splicing rules and symmetry principles. They are however constructed differently with modules of radically different shapes. This leads to a new perspective of treating them as tiling problems in topology. The field is open to explore further the dissection of squares and rectangles in relation to *kolam* designs. □

Acknowledgments.

I have relied heavily on T.V. Suresh for creating the U-tiles and motifs on the computer. I thank him for his help.

My brother Rangan Sundaresan has gifted me many books on mathematics and mathematicians, which have sustained my interest in mathematics as a hobby. A recent gift, the book by Professor Ian Stewart [1] is the inspiration for the present work – a generalization of Versatiles to Universatiles. Over the last decade my brother Srinivasan Sundaresan has hosted my website and has meticulously formatted all my papers for the site. His comments have contributed to improvement in the paper. I thank my brothers for their encouragement and support.

My daughter Venil Sumantran has helped me with useful suggestions, sustained encouragement and occasional prodding in all my *kolam* work. I am grateful to her.

Dedication.

This paper is dedicated to the memory of my daughter Vidya Ramanan. It is a much –needed distraction for me from her death three months ago. She had remarked "my father has always interesting things to say" – a rare compliment.

References and Notes.

[1] Ian Stewart (2014). "Professor Stewart's Casebook of Mathematical Mysteries" '*The Enigma of the Versatile Tile'* p 195, 292. (Basic Books, New York).

[2] S. Naranan. *Kolam* Designs Based on Fibonacci Numbers (Parts I to V). www.vindhiya.com/snaranan/fk/index.htm.

[3] S. Naranan. *Kolam* Designs Based on Fibonacci Numbers : an Overview. www.vindhiya.com/snaranan/fk/index.htm.

[4] A better choice for the grid is 10 x 13 that has side lengths 9 and 12.

[5] Martin Gardner (2001). The Colossal Book of Mathematics. (W.W. Norton & Company, New York, London).

[6] When rounded off to the nearest integer $(GR)^n/\sqrt{5}$ gives exactly F_n. E.g. $F_2=1$, $F_3=2$, $F_4=3$, $F_5=5$ and the approximation gives respectively 1.17, 1.89, 3.06, 4.95).

[7] The problem is to trace the evolution of loops as splices are added – in sets of 4 or 2 – starting from i loops (# of modules each single loop) and ending up in j loops. There is empirical evidence that 'parity is conserved' – i.e. i, j are both even or both odd. In FK's $i = 5$ (# of modules) and j can be 1. However in U-*kolam*s $i = 4$ and the minimum j value is 2.

List of Figures and Tables.

Figure 1. The Versatile

Figure 2. Arrangement of Versatile into four different motifs.

Figure 3. Same as Figure 2 with labels of measures (angles and lengths).

Figure 4. Versatile *Kolam* – square 10 x 10.

Figure 5. Versatile *Kolam* – square 12 x 12.

Figure 6. Versatile *Kolam* – rectangle 8 x 14.

Figure 7. Versatile *Kolam* – parallelogram 9 x 12.

Figure 8. Versatile *Kolam* -- rectangle 9 x 14.

Figure 9. Versatile *Kolam* – square 9 x 9.

Figure 10. Universatile ($\varphi = 30^o$) and arrangement into 4 motifs.

Figure 11a, 11b. The Fallacy of the missing square.

Figure 12. Eight Basic Shapes and 31 Distinct Orientations.

Figure 13. Fibonacci *Kolam* Construction.

Table 1. Universatiles: Arrangement of 4 U-tiles to form different Figures.

Table 2. Simple fractional approximations to functions of φ.

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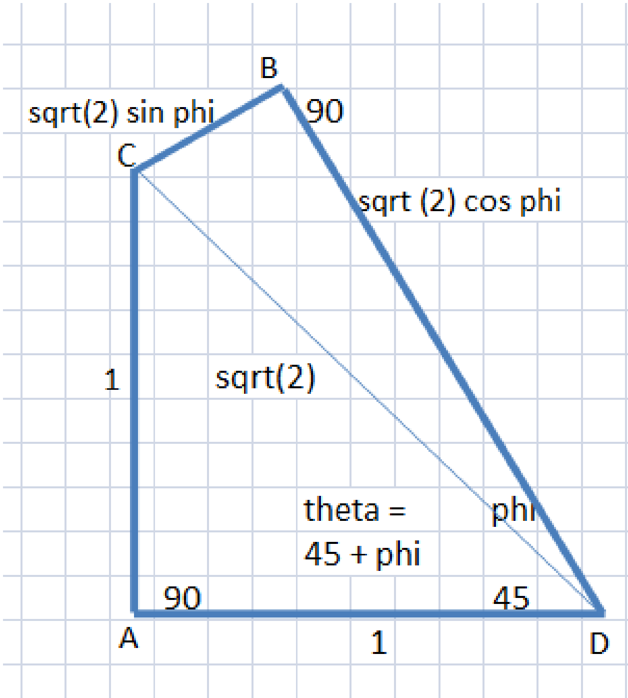
Figure 1



B
sqrt(2) sin phi
90
C
sqrt (2) cos phi
1
sqrt(2)
theta =
45 + phi
phi
90
45
A
1
D

Figure 2

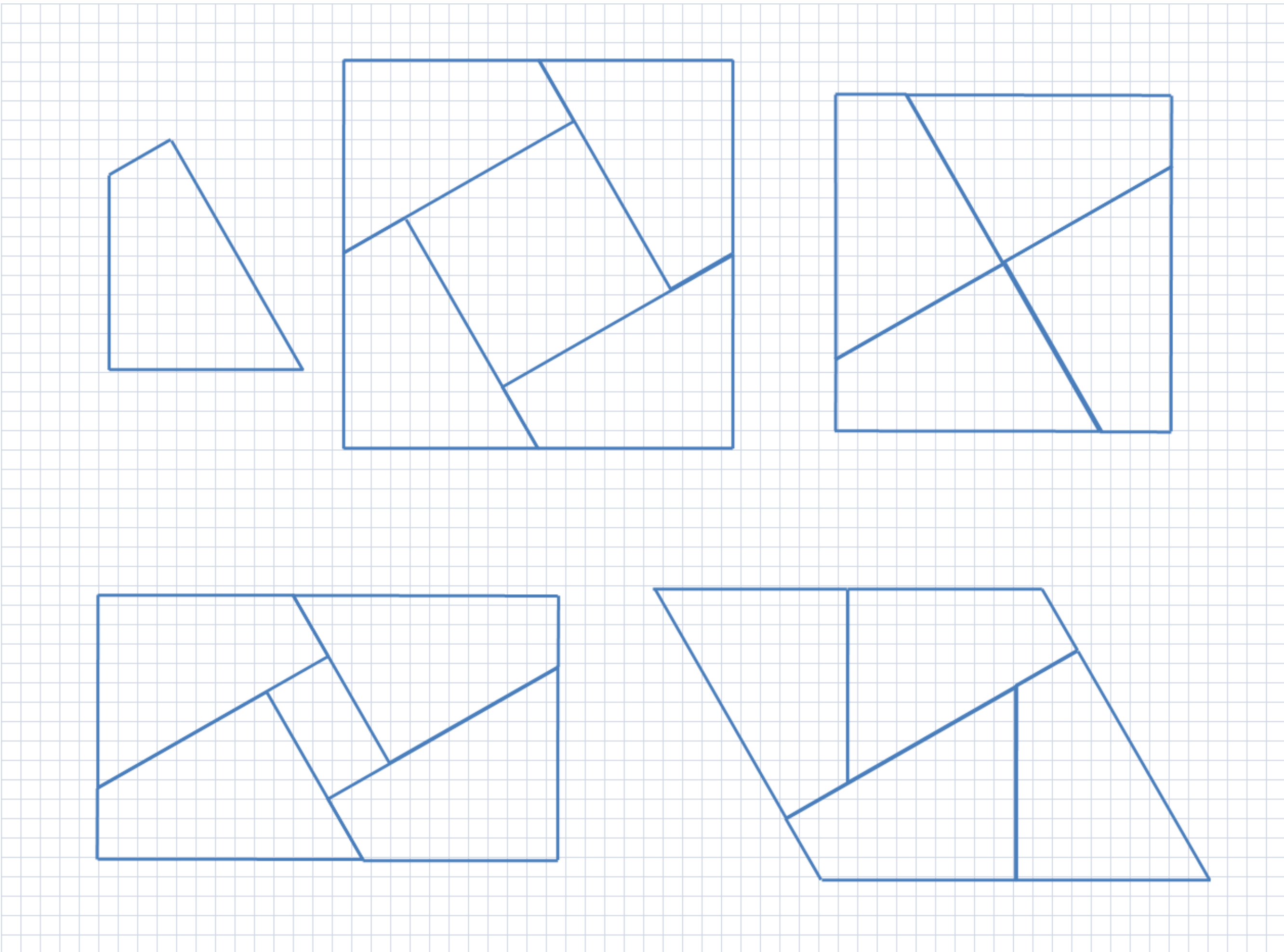
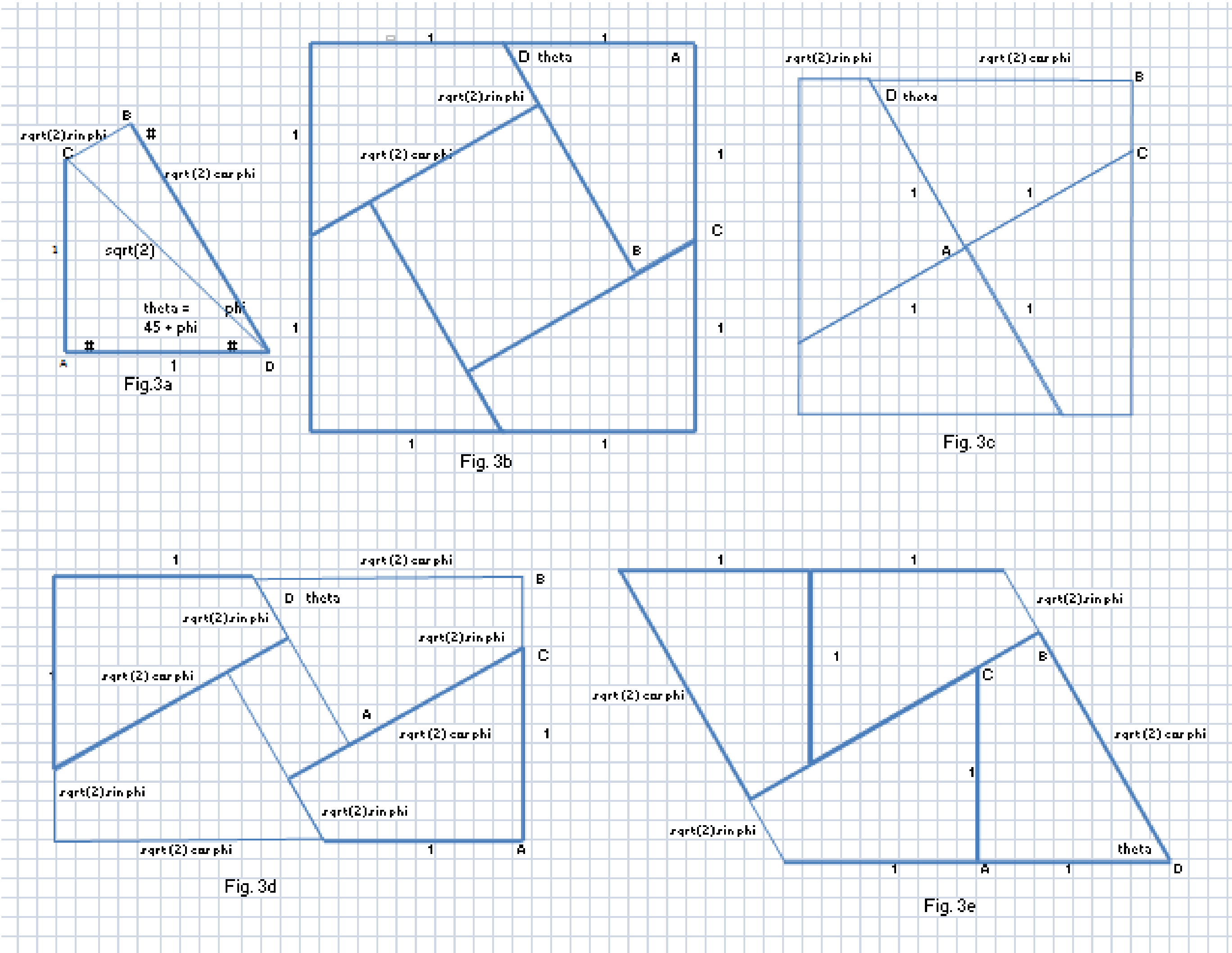


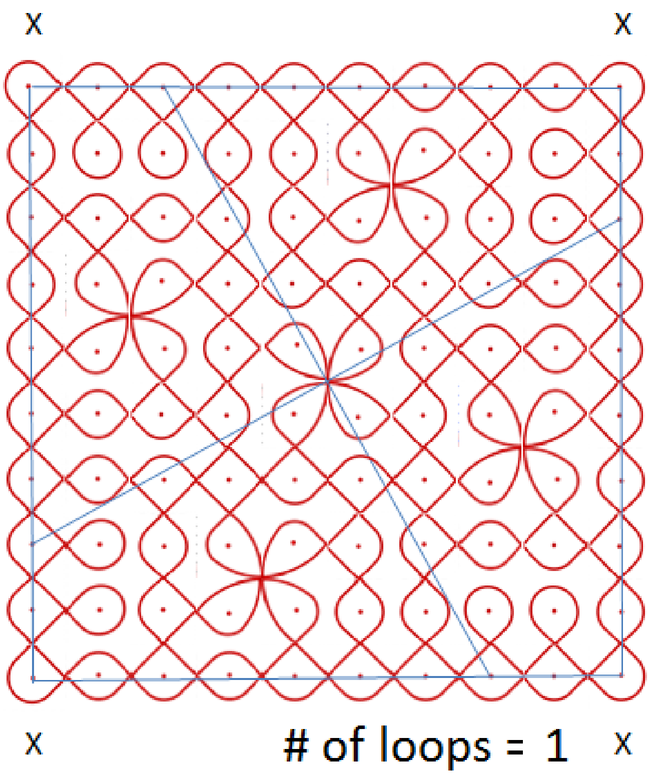
Figure 3



AC = AD =1   BC = Sqrt(2) sin phi   BD = Sqrt(2) cos phi   Angles A = B = 90 deg, C = 135 - phi, D = 45 + ph THETA = 45 + phi

Figure 4

Versatile 10 x 10



# of loops = 1

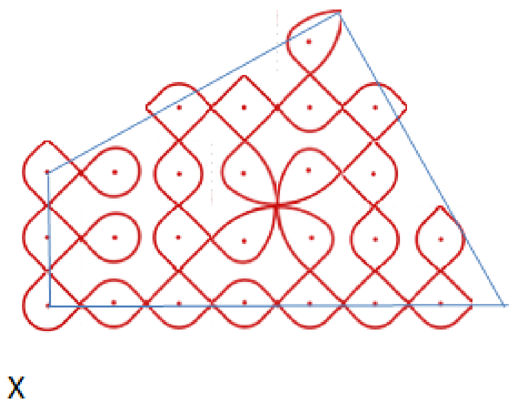
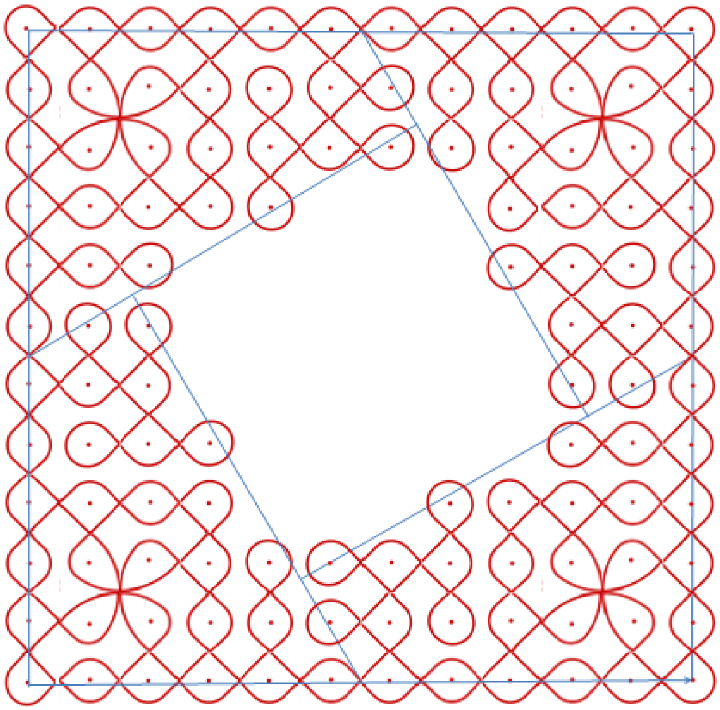
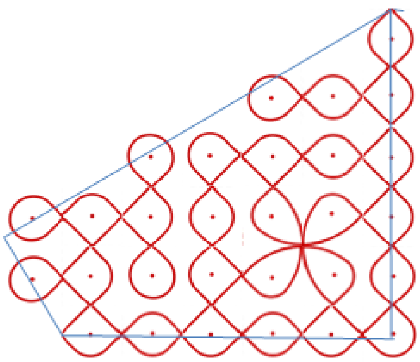


Figure 5

Versatile 12 x 12 square

#1  #2

# of loops 2 X



X

Figure 6

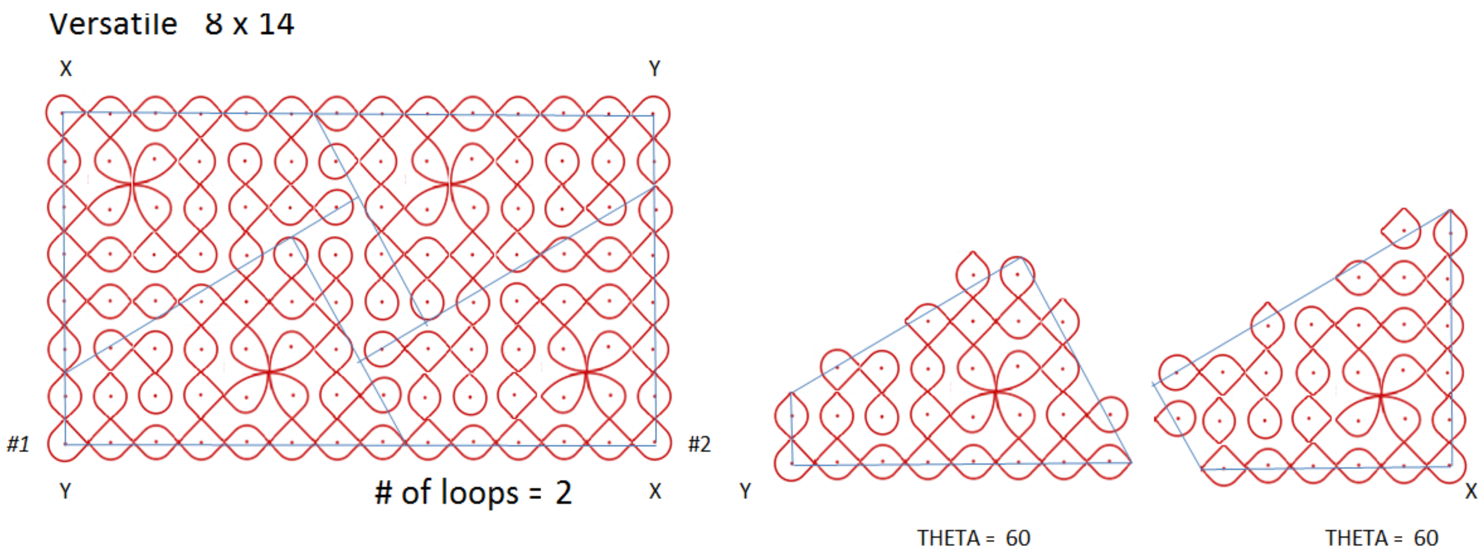


Figure 7

Versatile 9 x 12 Parallelogram

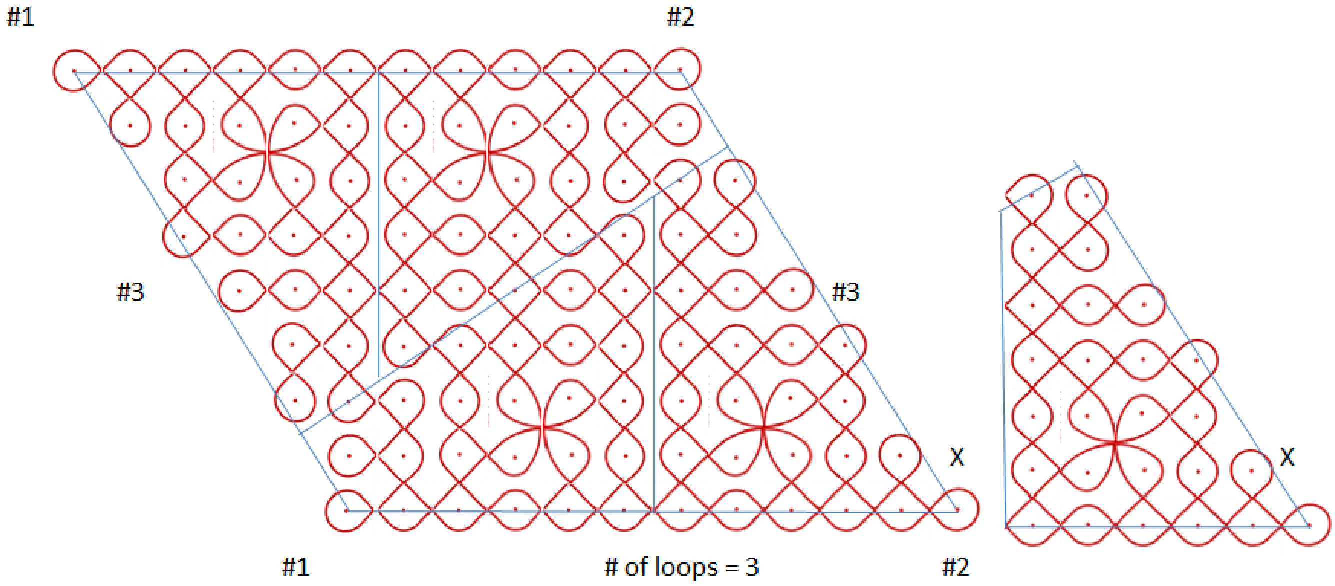


Figure 8

Versatile 9 x 14

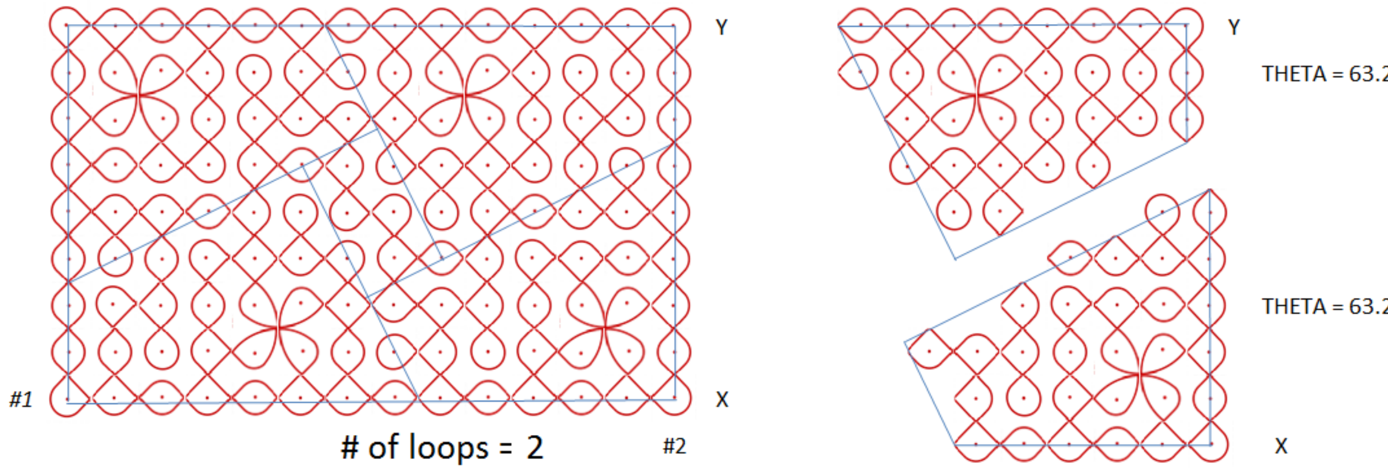


Figure 9

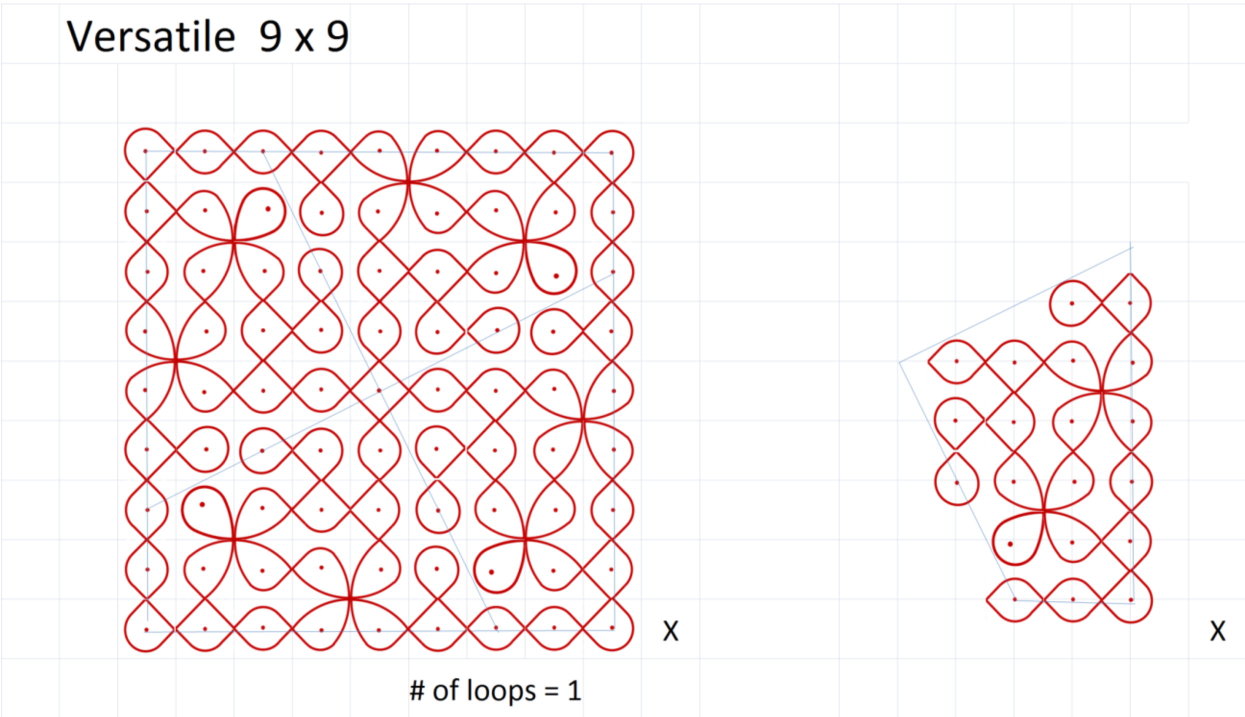


Figure 10

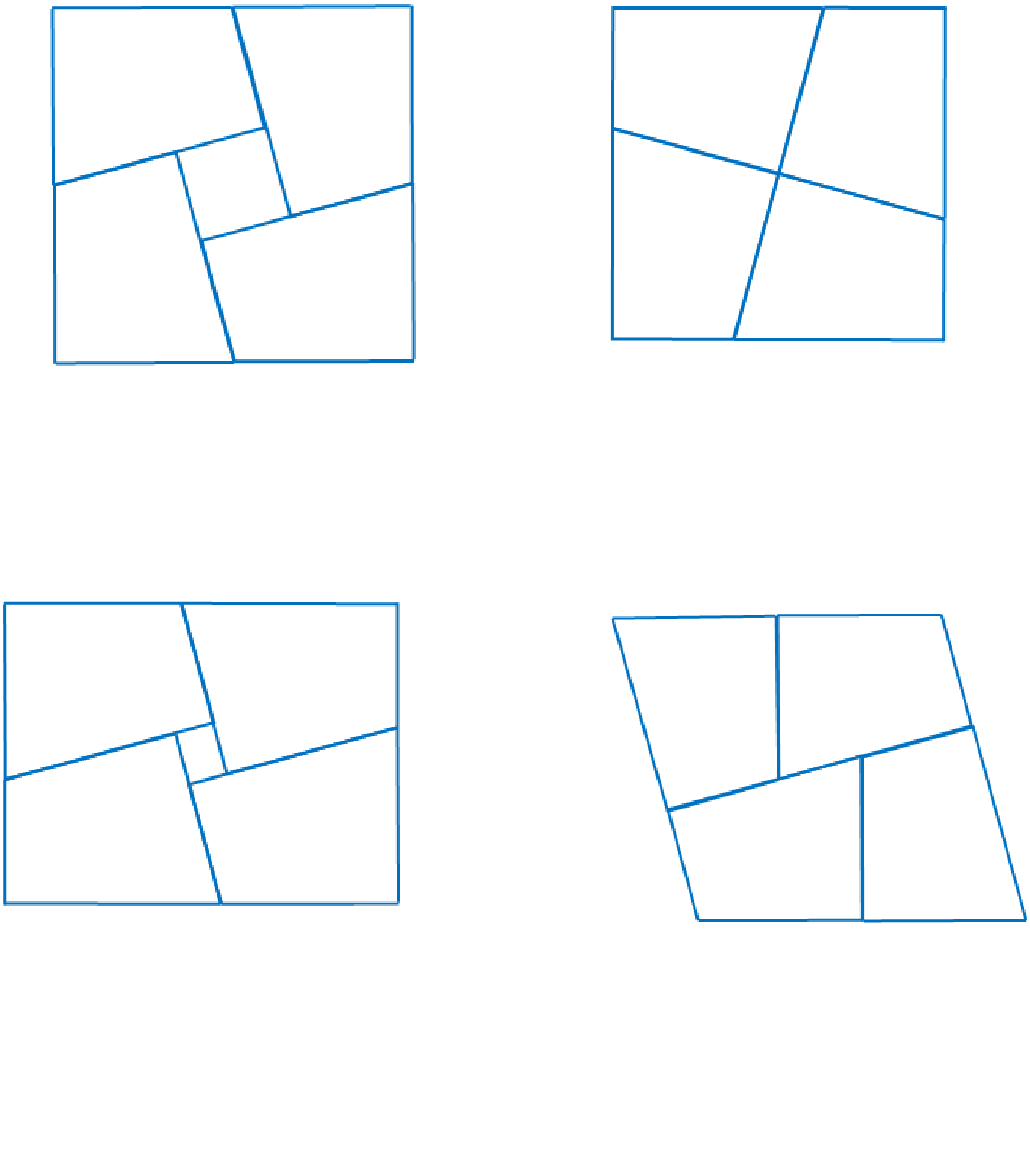


Figure 11a

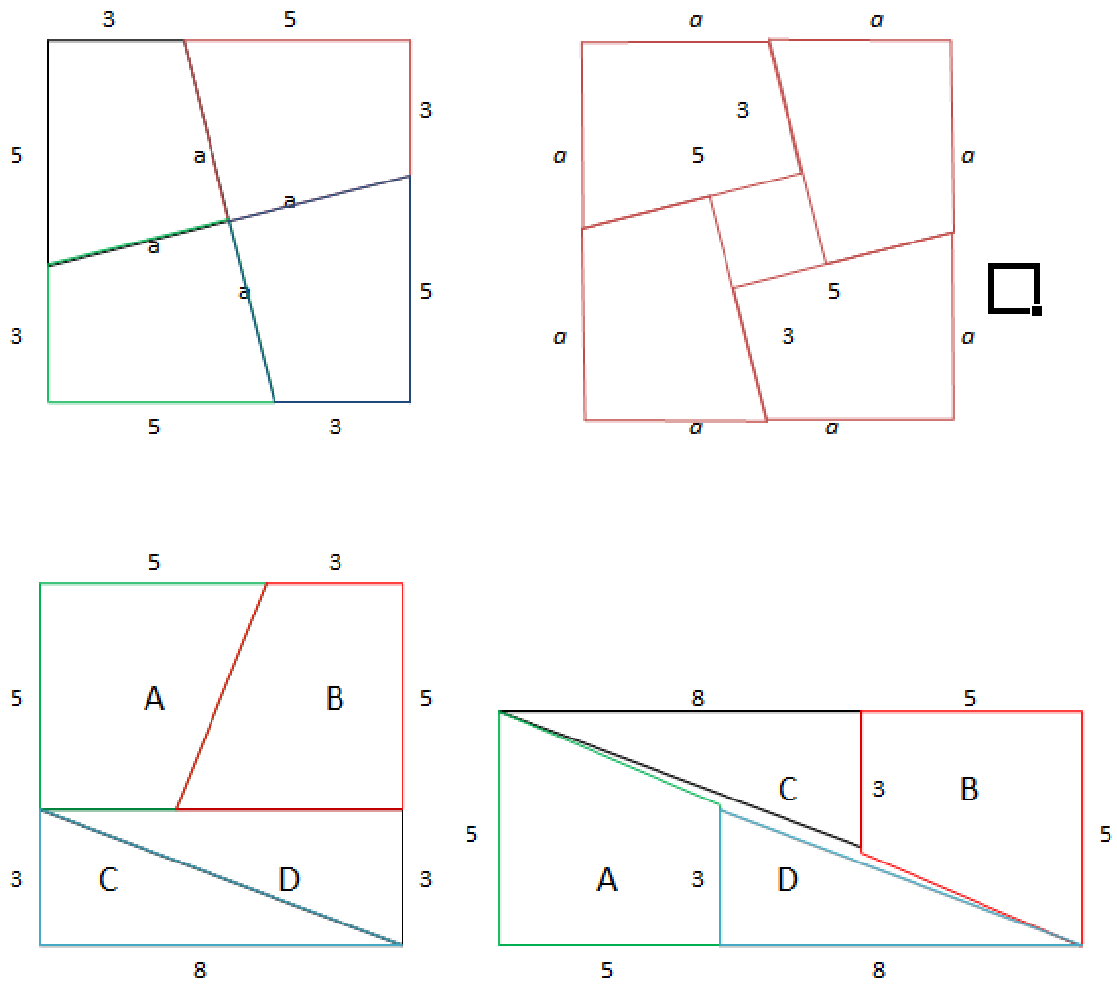


Figure 11b.

(In the left square, side is 8 units divided in ratio 4.5:3.5)

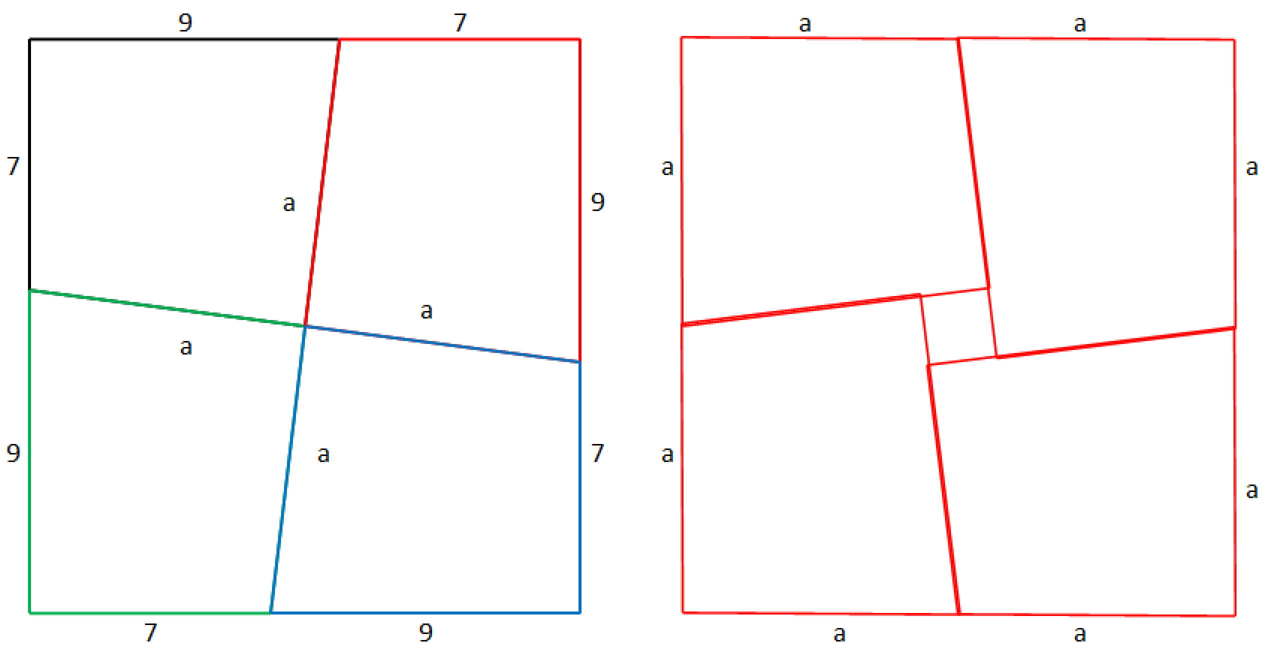
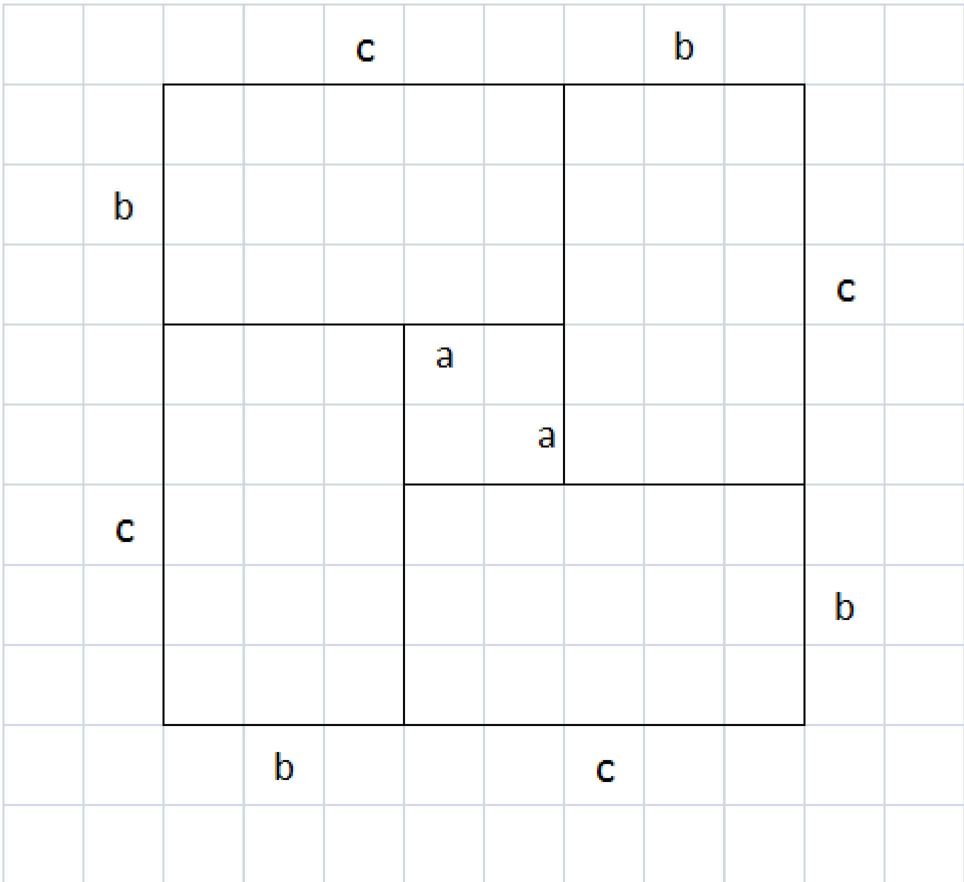


Figure 12

(Eight Basic Shapes and 31 distinct orientations)

Figure 13



$d = c + b \qquad a = c - b$

$dxd = axa + 4\,b\,c$

Table 1

UNIVERSATILES: ARRANGEMENTS OF FOUR U-TILES TO FORM DIFFERENT FIGURES

FIGURE DESCRIPTION	FIGURE	HEIGHT (h)	WIDTH (w)	RATIO (h/w)	AREA (h x w)
Square	3c	2 cos (45-ϕ) (*1.732*)	2 cos (45-ϕ) (*1.732*)	1 (*1*)	2 (1+ sin 2ϕ) (*3*)
Square with a hole					
Square	3b	2 (*2*)	2 (*2*)	1 (*1*)	4 (*4*)
Hole (square)		2 cos (45+ϕ) (*1*)	2 cos (45+ϕ) (*1*)	1 (*1*)	2 (1- sin 2ϕ) (*1*)
Rectangle with a hole					
Rectangle	3d	1+- √2 sin ϕ (*1.366*)	1 + √2 cos ϕ (*2.366*)	h/w (*0.577*)	2 cos (45-ϕ) +(1+sin 2ϕ) (*3.232*)
Hole (Rectangle)		√2 cos ϕ - 1 (*0.366*)	1 - √2 sin ϕ (*0.634*)	h/w (*0.577*)	2 cos (45-ϕ) -(1+sin 2ϕ) (*0.232*)
Parallelogram	3e	2 cos (45-ϕ) sin (45+ϕ) (*1.5*)	2 (*2*)	h/w (*0.750*)	4 cos (45-ϕ) sin (45+ϕ) (*3*)

The values in paranthesis (in italics) are calculated for phi = 15 deg

Table 2

Table 2. Simple fractional approximations to functions of phi

tan ϕ			√ 2 cos ϕ		√ 2sin ϕ	
Ratio	Fraction	ϕ	√ 2 cos ϕ	Fraction	√ 2 sin ϕ	Fraction
0.175	(1/6)	9.926	1.393	(7/5)	0.244	(1/4)
0.250	(1/4)	14.036	1.372	(11/8)	0.343	(1/3)
0.268	**(3/11)**	**15.000**	**1.366**	**(11/8)**	**0.366**	**(4/11)**
0.286	(2/7)	15.950	1.360	(4/3)	0.389	(2/5)
0.300	(3/10)	16.699	1.354	(4/3)	0.406	(2/5)
0.333	(1/3)	18.263	1.343	(4/3)	0.443	(4/9)
0.400	(2/5)	21.801	1.313	(4/3)	0.525	(1/2)
0.414	**(3/7)**	**22.500**	**1.306**	**(4/3)**	**0.541**	**(6/11)**
0.429	(3/7)	23.200	1.300	(4/3)	0.557	(5/9)
0.500	(1/2)	26.565	1.265	(5/4)	0.632	(5/8)
0.572	(4/7)	29.740	1.228	(11/9)	0.701	(7/10)
0.577	**(4/7)**	**30.000**	**1.225**	**(11/9)**	**0.707**	**(7/10)**
0.600	(3/5)	30.964	1.215	(6/5)	0.727	(5/7)
0.618	***(8/13)***	***31.716***	***1.203***	***(6/5)***	***0.743***	***(3/4)***
0.700	(7/10)	34.992	1.158	(8/7)	0.811	(4/5)
0.800	(4/5)	38.650	1.104	(11/10)	0.883	(8/9)

Numbers in bold italics correspond to the Golden Ratio 0.618